

# The Pointwise Estimates of Solutions for Semilinear Dissipative Wave Equation

Yongqin Liu\*

*Department of Mathematics, Fudan University, Shanghai, China*

*Department of Mathematics, Kyushu University, Fukuoka, Japan*

## Abstract

In this paper we focus on the global-in-time existence and the pointwise estimates of solutions to the initial value problem for the semilinear dissipative wave equation in multi-dimensions. By using the method of Green function combined with the energy estimates, we obtain the pointwise decay estimates of solutions to the problem.

**keywords:** semilinear dissipative wave equation, pointwise estimates, Green function.

**MSC(2000):** 35E15; 35L15.

## 1 Introduction

In this paper we consider the initial value problem for the semilinear dissipative wave equation in  $n(n \geq 1)$  dimensions,

$$(\square + \partial_t)u(x, t) = f(u), \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1.1) \quad \boxed{1a}$$

with initial condition

$$(u, \partial_t u)(x, 0) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n, \quad (1.2) \quad \boxed{IC}$$

where  $\square + \partial_t = \partial_t^2 - \Delta_x + \partial_t$  is the dissipative wave operator with Laplacian  $\Delta_x = \sum_{j=1}^n \partial_{x_j}^2$ ,  $f(u) = -|u|^\theta u$ ,  $\theta > 0$  is an integer. Equation (1.1) is often called the semilinear dissipative wave equation or semilinear telegraph equation.

There have been many results on the equation (1.1) and its variants corresponding to the different forms of  $f(u)$ . By employing the weighted  $L^2$

---

\*email: yqliu2@yahoo.com.cn

energy method and the explicit formula of solutions, Ikehata, Nishihara and Zhao [7] obtained that the behavior of solutions to (1.1) as  $t \rightarrow \infty$  is expected to be same as that for the corresponding heat equation, Nishihara [15] studied the global asymptotic behaviors in three and four dimensions, and Nishihara and Zhao [19] obtained the decay properties of solutions to the problem (1.1)(1.2). Kawashima, Nakao and Ono [9] studied the decay property of solutions to (1.1) by using the energy method combined with  $L^p - L^q$  estimates, and Ono [20] derived sharp decay rates in the subcritical case of solutions to (1.1) in unbounded domains in  $\mathbb{R}^N$  without the smallness condition on initial data. Also, recently Nishihara, etc. in [16, 17] studied the following semilinear damped wave equations with time or space-time dependent damping term,

$$u_{tt} - \Delta u + b(t)u_t + |u|^{\rho-1}u = 0, \quad (1.3) \quad \boxed{n1}$$

and

$$u_{tt} - \Delta u + b(t, x)u_t + |u|^{\rho-1}u = 0, \quad (1.4) \quad \boxed{n2}$$

where  $\rho > 1$ ,  $b(t) = b_0(1+t)^{-\beta}$  with  $b_0 > 0, -1 < \beta < 1$ , and  $b(t, x) = b_0(1+|x|^2)^{-\frac{\alpha}{2}}(1+t)^{-\beta}$  with  $b_0 > 0, \alpha \geq 0, \beta \geq 0, \alpha + \beta \in [0, 1)$ , and obtained the global existence and the  $L^2$  decay rate of the solution by using the weighted energy method. (1.3) and (1.4) with the exponents  $\alpha = \beta = 0$  yield (1.1). For studies on the case  $f(u) = |u|^\theta u$ , see [3, 6, 8, 13, 14, 18], for studies on the case  $f(u) = |u|^{\theta+1}$ , see [5, 12, 21, 22, 23, 25], and for studies on the global attractors, see [1, 10] and the references cited there.

The main purpose of this paper is to study the pointwise estimates of solutions for (1.1)(1.2). In [11], Liu and Wang studied the corresponding linear problem, i.e. (1.1) with  $f(u) = 0$  and (1.2), and obtained the pointwise estimates of solutions. In this paper, we first obtain the global-in-time solutions by energy method combined with the fixed point theorem of Banach, and then obtain the optimal pointwise decay estimates of the solutions by using the properties of the Green function proved in [11] combined with Fourier analysis. One point worthy to be mentioned is that, different from that for solutions to the corresponding linear problem, the order of derivatives with respect to time variable  $t$  of solutions does not contribute to the decay rate of solutions due to the presence of the semilinear term, which could be seen from (2.4) in Theorem 2.4 and (5.1) in Theorem 5.1.

The rest of the paper is arranged as follows. In section 2, the main results are stated. We give the proof of Proposition 2.3 and then obtain the global-in-time existence of solutions in section 3. In section 4 we give estimates on the Green function by Fourier analysis which will be used in the last section where the proof of Theorem 2.4 is given.

Before closing this section, we give some notations to be used below. Let  $\mathcal{F}[f]$  denote the Fourier transform of  $f$  defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and we denote its inverse transform by  $\mathcal{F}^{-1}$ .

For  $1 \leq p \leq \infty$ ,  $L^p = L^p(\mathbb{R}^n)$  is the usual Lebesgue space with the norm  $\|\cdot\|_{L^p}$ . Let  $s$  be a nonnegative integer. Then  $H^s = H^s(\mathbb{R}^n)$  denotes the Sobolev space of  $L^2$  functions, equipped with the norm

$$\|f\|_{H^s} := \left( \sum_{k=0}^s \|\partial_x^k f\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

In particular, we use  $\|\cdot\| = \|\cdot\|_{L^2}$ ,  $\|\cdot\|_s = \|\cdot\|_{H^s}$ . Here, for a multi-index  $\alpha$ ,  $D_x^\alpha$  denotes the totality of all the  $|\alpha|$ -th order derivatives with respect to  $x \in \mathbb{R}^n$ . Also,  $C^k(I; H^s(\mathbb{R}^n))$  denotes the space of  $k$ -times continuously differentiable functions on the interval  $I$  with values in the Sobolev space  $H^s = H^s(\mathbb{R}^n)$ .

Finally, in this paper, we denote every positive constant by the same symbol  $C$  or  $c$  without confusion.  $[\cdot]$  is Gauss' symbol.

## 2 Main theorems

The first main result is about the global existence of solutions to the initial value problem  $(\text{Ia})(\text{IC})$ .

**ge** **Theorem 2.1** (Global existence). *Let  $\theta > 0$  be an integer. Assume that  $(u_0, u_1) \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ ,  $s \geq [\frac{n}{2}]$ , put*

$$E_0 := \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s}.$$

*Then if  $E_0$  is suitably small,  $(\text{Ia})(\text{IC})$  admits a unique solution*

$$u \in \bigcap_{i=0}^{s+1} C^i([0, \infty); H^{s+1-i}(\mathbb{R}^n)),$$

*which satisfies*

$$\sum_{i=0}^{s+1} \|\partial_t^i u(t)\|_{s+1-i}^2 + \int_0^t (\|\nabla u(\tau)\|_s^2 + \sum_{i=1}^{s+1} \|\partial_\tau^i u(\tau)\|_{s+1-i}^2) d\tau \leq C E_0^2. \quad (2.1) \quad \text{ge1}$$

Theorem <sup>ge</sup>2.1 is proved by combining the local existence of solutions stated in the following Theorem <sup>le</sup>2.2 with a priori estimate in the following Proposition <sup>ae</sup>2.3.

**1e** **Theorem 2.2** (Local existence). *Let  $\theta > 0$  be an integer. Assume that  $(u_0, u_1) \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ ,  $s \geq [\frac{n}{2}]$ , then there exists  $T > 0$  and a unique solution to <sup>1a</sup>(1.1) <sup>ic</sup>(1.2) satisfying*

$$u \in \bigcap_{i=0}^{s+1} C^i([0, T]; H^{s+1-i}(\mathbb{R}^n)).$$

The proof of the local existence result is based on the fixed point theorem of Banach and standard argument, so the detail is omitted.

Based on the a priori assumption

$$\sup_{0 < t < T} \|u(t)\|_{L^\infty} \leq \bar{\delta}, \quad (2.2) \quad \text{aa}$$

where  $s > n$  is an integer and  $\bar{\delta} < 1$  is a small constant, the following a priori estimate is obtained.

**ae** **Proposition 2.3** (A priori estimate). *Under the same assumptions as in Theorem <sup>ge</sup>2.1, let  $u(x, t)$  be the solution to <sup>1a</sup>(1.1) <sup>ic</sup>(1.2) which is defined on  $[0, T]$  and verifies <sup>aa</sup>(2.2), then the following estimate holds,*

$$\sup_{0 < t < T} \left\{ \sum_{i=0}^{s+1} \|\partial_t^i u(t)\|_{s+1-i}^2 + \int_0^T (\|\nabla u(\tau)\|_s^2 + \sum_{i=1}^{s+1} \|\partial_\tau^i u(\tau)\|_{s+1-i}^2) d\tau \right\} \leq CE_0^2. \quad (2.3) \quad \text{ae1}$$

**Remark 1.** In <sup>1a</sup>(1.1),  $f(u) = -|u|^\theta u$  is called absorption term which makes it possible to close energy estimates. Otherwise, if  $f(u) = |u|^\theta u$ , then Theorem <sup>ge</sup>2.1 does not hold, since the lower-order term present in the energy estimates could not be controlled.

The second main result is about the pointwise estimate to the solution obtained in Theorem <sup>ge</sup>2.1.

**pe** **Theorem 2.4** (Pointwise estimate). *Under the same assumptions as in Theorem <sup>ge</sup>2.1, if  $s > n$ ,  $\theta \geq 2 + [\frac{1}{n}]$ , and for any multi-indexes  $\alpha$ ,  $|\alpha| < s - \frac{n}{2}$ , there exists some constant  $r > \max\{\frac{n}{2}, 1\}$  such that*

$$|D_x^\alpha u_0(x)| + |D_x^\alpha u_1(x)| \leq C(1 + |x|^2)^{-r},$$

then for  $h \geq 0$  satisfying  $|\alpha| + h < s - n$ , the solution to <sup>1a</sup>(1.1) <sup>ic</sup>(1.2) obtained in Theorem <sup>ge</sup>2.1 satisfies the following pointwise estimate,

$$|\partial_t^h D_x^\alpha u(x, t)| \leq CE_0(1 + t)^{-\frac{n+|\alpha|}{2}} (1 + \frac{|x|^2}{1+t})^{-r}. \quad (2.4) \quad \text{pe1}$$

**Remark 2.** From the estimate in Theorem <sup>pe</sup>2.4, we see that the order of derivatives with respect to  $t$  of the solution obtained in Theorem <sup>ge</sup>2.1 has no effect on the decay rate of the solution, as is different from that for the solution to the corresponding linear problem studied in <sup>lw</sup>[11].

As a direct corollary of Theorem <sup>pe</sup>2.4 we have

44 **Corollary 2.5.** Assume that the same assumptions as in Theorem <sup>pe</sup>2.4 hold, then for  $p \in [1, \infty]$ ,  $|\alpha| + h < s - n$ , the solution to <sup>la</sup>(1.1) <sup>lc</sup>(1.2) satisfies,

$$\|\partial_t^h D_x^\alpha u(\cdot, t)\|_{L^p} \leq C E_0 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}.$$

### 3 The global existence of solutions

First we give a lemma which will be used in our next energy estimates.

L **Lemma 3.1.** Let  $n \geq 1$ ,  $1 \leq p, q, r \leq \infty$  and  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ . Then the following estimate holds:

$$\|\partial_x^k(uv)\|_{L^p} \leq C(\|u\|_{L^q} \|\partial_x^k v\|_{L^r} + \|v\|_{L^q} \|\partial_x^k u\|_{L^r}) \quad (3.1) \quad \text{d1}$$

for  $k \geq 0$ .

*Proof.* The estimate <sup>g1</sup>(3.1) can be found in a literature but we give here a proof. To prove <sup>g1</sup>(3.1), it is enough to show that, for  $k_1 \geq 1$ ,  $k_2 \geq 1$  and  $k_1 + k_2 = k$ , the following estimate holds:

$$\|\partial_x^{k_1} u \partial_x^{k_2} v\|_{L^p} \leq C(\|u\|_{L^q} \|\partial_x^{k_1} v\|_{L^r} + \|v\|_{L^q} \|\partial_x^{k_1} u\|_{L^r}).$$

Let  $\theta_j = \frac{k_j}{k}$ ,  $j = 1, 2$ , and define  $p_j$ ,  $j = 1, 2$ , by

$$\frac{1}{p_j} - \frac{k_j}{n} = (1 - \theta_j) \frac{1}{q} + \theta_j \left( \frac{1}{r} - \frac{k}{n} \right).$$

Since  $\theta_1 + \theta_2 = 1$ , we have  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ . By using the Hölder inequality and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \|\partial_x^{k_1} u \partial_x^{k_2} v\|_{L^p} &\leq \|\partial_x^{k_1} u\|_{L^{p_1}} \|\partial_x^{k_2} v\|_{L^{p_2}} \\ &\leq C(\|u\|_{L^q}^{1-\theta_1} \|\partial_x^{k_1} u\|_{L^r}^{\theta_1}) (\|v\|_{L^q}^{1-\theta_2} \|\partial_x^{k_2} v\|_{L^r}^{\theta_2}) \\ &\leq C(\|u\|_{L^q} \|\partial_x^{k_1} v\|_{L^r})^{\theta_2} (\|v\|_{L^q} \|\partial_x^{k_1} u\|_{L^r})^{\theta_1} \\ &\leq C(\|u\|_{L^q} \|\partial_x^{k_1} v\|_{L^r} + \|v\|_{L^q} \|\partial_x^{k_1} u\|_{L^r}). \end{aligned}$$

In the last inequality, we have used the Young inequality. Thus <sup>g1</sup>(3.1) is proved. □

Now, let  $T > 0$  and consider solutions to the problem  $(\text{I.1})$ ,  $(\text{I.2})$ , which are defined on the time interval  $[0, T]$  and verify the regularity mentioned in Proposition  $\text{2.3}$ . we derive energy estimates under the a priori assumption  $(\text{2.2})$ .

By multiplying  $(\text{I.1})$  with  $u_t$  and integrating on  $\mathbb{R}^n \times (0, t)$  with respect to  $(x, t)$ , we get

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \int_{\mathbb{R}^n} |u|^{\theta+2}(x, t) dx + \int_0^t \|u_\tau(\tau)\|^2 d\tau \leq CE_0^2. \quad (3.2) \quad \boxed{3a0}$$

$\forall \alpha, 1 \leq |\alpha| \leq s$ , by multiplying  $D_x^\alpha(\text{I.1})$  with  $D_x^\alpha u_t$  and integrating on  $\mathbb{R}^n \times (0, t)$  with respect to  $(x, t)$ , in view of Lemma  $\text{3.1}$  and  $(\text{2.2})$  we get

$$\begin{aligned} & \|D_x^\alpha u_t(t)\|^2 + \|D_x^\alpha \nabla u(t)\|^2 + \int_0^t \|D_x^\alpha u_\tau(\tau)\|^2 d\tau \\ & \leq CE_0^2 + C \int_0^t \|u(\tau)\|_{L^\infty}^{2\theta} \|D_x^\alpha u(\tau)\|^2 d\tau \\ & \leq CE_0^2 + C\bar{\delta} \int_0^t \|D_x^\alpha u(\tau)\|^2 d\tau. \end{aligned}$$

By taking sum for  $\alpha$  with  $1 \leq |\alpha| \leq s$ , it yields that

$$\begin{aligned} & \|\nabla u_t(t)\|_{s-1}^2 + \|\nabla^2 u(t)\|_{s-1}^2 + \int_0^t \|\nabla u_\tau(\tau)\|_{s-1}^2 d\tau \\ & \leq CE_0^2 + C\bar{\delta} \int_0^t \|\nabla u(\tau)\|_{s-1}^2 d\tau. \end{aligned} \quad (3.3) \quad \boxed{3a}$$

By multiplying  $(\text{I.1})$  with  $u$  and integrating on  $\mathbb{R}^n \times (0, t)$  with respect to  $(x, t)$ , by virtue of  $(\text{3a0})$  we get

$$\begin{aligned} & \|u(t)\|^2 + \int_0^t \|\nabla u(\tau)\|^2 d\tau + \int_0^t \int_{\mathbb{R}^n} (|u|^{\theta+2})(x, \tau) dx d\tau \\ & \leq C(E_0^2 + \|u_t(t)\|^2 + \int_0^t \|u_\tau(\tau)\|^2 d\tau) \leq CE_0^2. \end{aligned} \quad (3.4) \quad \boxed{3b}$$

$\forall \alpha, 1 \leq |\alpha| \leq s$ , by multiplying  $D_x^\alpha(\text{I.1})$  with  $D_x^\alpha u$  and integrating on  $\mathbb{R}^n \times (0, t)$  with respect to  $(x, t)$ , by virtue of Lemma  $\text{3.1}$  we get

$$\begin{aligned} & \|D_x^\alpha u(t)\|^2 + \int_0^t \|D_x^\alpha \nabla u(\tau)\|^2 d\tau \leq C(E_0^2 + \|D_x^\alpha u_t(t)\|^2 + \int_0^t \|D_x^\alpha u_\tau(\tau)\|^2 d\tau) \\ & + C \int_0^t \|u(\tau)\|_{L^\infty}^{2\theta} \|D_x^\alpha u(\tau)\|^2 d\tau. \end{aligned}$$

By taking sum for  $\alpha$  with  $1 \leq |\alpha| \leq s$  and in view of  $(\text{3a})$  and  $(\text{2.2})$ , it yields that

$$\|\nabla u(t)\|_{s-1}^2 + \int_0^t \|\nabla^2 u(\tau)\|_{s-1}^2 d\tau \leq CE_0^2 + C\bar{\delta} \int_0^t \|\nabla u(\tau)\|_{s-1}^2 d\tau. \quad (3.5) \quad \boxed{3c}$$

(3a0), (3a), (3b) and (3c) yield that

$$\|u(t)\|_{s+1}^2 + \|u_t(t)\|_s^2 + \int_0^t (\|\nabla u(\tau)\|_s^2 + \|u_\tau(\tau)\|_s^2) d\tau \leq CE_0^2. \quad (3.6) \quad \boxed{3d}$$

*Proof of Proposition 2.3.* To prove (2.3) in Proposition 2.3, it is enough to prove that the following estimate holds for  $\forall h \in [1, s+1]$ ,  $\forall t \in [0, T]$ .

$$\sum_{i=0}^h \|\partial_t^i u(t)\|_{s+1-i}^2 + \int_0^t (\|\nabla u(\tau)\|_s^2 + \sum_{i=1}^h \|\partial_\tau^i u(\tau)\|_{s+1-i}^2) d\tau \leq CE_0^2. \quad (3.7) \quad \boxed{3e}$$

It is obvious that (3.7) holds with  $h = 1$  by virtue of (3d). Assume that (3.7) holds with  $h = j$  ( $1 \leq j \leq s$ ), next we will prove that (3.7) holds with  $h = j+1$ .

From (1.1), by using induction argument we could prove that the following two equalities hold for  $k \geq 1$ ,

$$\begin{aligned} \partial_t^{2k} u(x, t) = & a_{2k} \Delta^k u(x, t) + b_{2k} \Delta^{k-1} u_t(x, t) \\ & + P\{\Delta^i u(x, t), \Delta^j u_t(x, t), 0 \leq i \leq k-1, 0 \leq j \leq k-2\}, \end{aligned} \quad (3.8) \quad \boxed{\text{even}}$$

$$\begin{aligned} \partial_t^{2k+1} u(x, t) = & a_{2k+1} \Delta^k u(x, t) + b_{2k+1} \Delta^k u_t(x, t) \\ & + P\{\Delta^i u(x, t), \Delta^j u_t(x, t), 0 \leq i \leq k-1, 0 \leq j \leq k-1\}, \end{aligned} \quad (3.9) \quad \boxed{\text{odd}}$$

where  $a_{2k}$ ,  $a_{2k+1}$ ,  $b_{2k}$ ,  $b_{2k+1}$  are constants,  $P\{\Delta^i u(x, t), \Delta^j u_t(x, t), 0 \leq i \leq k-1, 0 \leq j \leq k-2\}$  is a polynomial with arguments  $\Delta^i u(x, t), \Delta^j u_t(x, t), 0 \leq i \leq k-1, 0 \leq j \leq k-2$ .

Let  $t = 0$  in (3.8)(3.9), we have for  $k \geq 1$ ,

$$\begin{aligned} \partial_t^{2k} u(x, 0) = & a_{2k} \Delta^k u_0(x) + b_{2k} \Delta^{k-1} u_1(x) \\ & + P\{\Delta^i u_0(x), \Delta^j u_1(x), 0 \leq i \leq k-1, 0 \leq j \leq k-2\}, \end{aligned} \quad (3.10) \quad \boxed{3h}$$

$$\begin{aligned} \partial_t^{2k+1} u(x, 0) = & a_{2k+1} \Delta^k u_0(x) + b_{2k+1} \Delta^k u_1(x) \\ & + P\{\Delta^i u_0(x), \Delta^j u_1(x), 0 \leq i \leq k-1, 0 \leq j \leq k-1\}. \end{aligned} \quad (3.11) \quad \boxed{3i}$$

$\forall \alpha, |\alpha| \leq s-j$ , by multiplying  $D_x^\alpha \partial_t^j$  (1.1) with  $D_x^\alpha \partial_t^{j+1} u$  and integrating on  $\mathbb{R}^n \times (0, t)$  with respect to  $(x, t)$ , in view of (3.7) with  $h = j$ , (3.10) and (3.11) we get

$$\|D_x^\alpha \partial_t^{j+1} u(t)\|^2 + \int_0^t \int_{\mathbb{R}^n} \|D_x^\alpha \partial_\tau^{j+1} u(\tau)\|^2 d\tau \leq CE_0^2.$$

By taking sum for  $\alpha$  with  $0 \leq |\alpha| \leq s - j$ , it yields that

$$\|\partial_t^{j+1}u(t)\|_{s-j}^2 + \int_0^t \|\partial_\tau^{j+1}u(\tau)\|_{s-j}^2 d\tau \leq CE_0^2. \quad (3.12) \quad \boxed{3f}$$

(3.7) with  $h = j$  and (3.12) yield that

$$\sum_{i=0}^{j+1} \|\partial_t^i u(t)\|_{s+1-i}^2 + \int_0^t (\|\nabla u(\tau)\|_s^2 + \sum_{i=1}^{j+1} \|\partial_\tau^i u(\tau)\|_{s+1-i}^2) d\tau \leq CE_0^2. \quad (3.13) \quad \boxed{3g}$$

It means that (3.7) holds with  $h = j + 1$ . Thus by induction method, we complete the proof of Proposition 2.3.  $\square$

Now we give the proof of Theorem 2.1.

*Proof of Theorem 2.1.* By virtue of the a priori estimate (2.3) in Proposition 2.3, we can continue a unique solution obtained in Theorem 2.2 globally in time, provided that  $E_0$  is suitably small, say,  $E_0 < \delta_0$ ,  $\delta_0$  depends only on  $\bar{\delta}$  in (2.2). The global solution thus obtained satisfies (2.1) and (2.2). This finishes the proof of Theorem 2.1.  $\square$

## 4 Estimates on Green function

In this section, we list some formulas and properties of the Green function obtained in [11] to make preparation for the next section about the pointwise estimates of solutions.

The Green function or the fundamental solution to the corresponding linear dissipative wave equation (i.e.  $f(u) = 0$  in (1.1)) to (1.1) satisfies

$$\begin{cases} (\square + \partial_t)G(x, t) = 0, & x \in \mathbb{R}^n, t > 0, \\ G(x, 0) = 0, & x \in \mathbb{R}^n, \\ \partial_t G(x, 0) = \delta(x), & x \in \mathbb{R}^n. \end{cases}$$

By Fourier transform we get that,

$$\begin{cases} (\partial_t^2 + \partial_t)\hat{G}(\xi, t) + |\xi|^2 \hat{G}(\xi, t) = 0, & \xi \in \mathbb{R}^n, t > 0, \\ \hat{G}(\xi, 0) = 0, & \xi \in \mathbb{R}^n, \\ \partial_t \hat{G}(\xi, 0) = 1, & \xi \in \mathbb{R}^n. \end{cases}$$



The symbol of the operator for equation (1.1) is

$$\sigma(\square + \partial_t) = \tau^2 + \tau + |\xi|^2, \quad (4.1) \quad \boxed{2a}$$

$\tau$  and  $\xi$  correspond to  $\frac{\partial}{\partial t}$ , and  $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$ ,  $j = 1, 2, \dots, n$ . It is easy to see that the eigenvalues of (4.1) are  $\tau = \mu_{\pm}(\xi) = \frac{1}{2}(-1 \pm \sqrt{1 - 4|\xi|^2})$ . By direct calculation we have that

$$\hat{G}(\xi, t) = (1 - 4|\xi|^2)^{-\frac{1}{2}} (e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}).$$

For convenience we decompose  $\hat{G}(\xi, t) = \hat{G}^+(\xi, t) + \hat{G}^-(\xi, t)$ , where

$$\hat{G}^{\pm}(\xi, t) = \pm \mu_0^{-1} e^{\mu_{\pm}(\xi)t}, \quad \mu_0(\xi) = (1 - 4|\xi|^2)^{\frac{1}{2}}.$$

Let

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| < \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases} \quad \chi_3(\xi) = \begin{cases} 1, & |\xi| > R, \\ 0, & |\xi| < R - 1, \end{cases}$$

be the smooth cut-off functions, where  $\varepsilon$  and  $R$  are any fixed positive numbers satisfying  $2\varepsilon < R - 1$ .

Set

$$\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi),$$

and

$$\hat{G}_i^{\pm}(\xi, t) = \chi_i(\xi) \hat{G}^{\pm}(\xi, t), \quad i = 1, 2, 3.$$

We are going to study  $G_i^{\pm}(x, t)$ , which is the inverse Fourier transform corresponding to  $\hat{G}_i^{\pm}(\xi, t)$ .

Denote  $B_N(|x|, t) = (1 + \frac{|x|^2}{1+t})^{-N}$ . First we give two propositions regarding to  $G_1(x, t)$  and  $G_2(x, t)$ , the proof can be seen in [11, 24].

**[21] Proposition 4.1.** *For sufficiently small  $\varepsilon$ , there exists constant  $C > 0$ , and  $N > n$  such that*

$$|\partial_t^h D_x^\alpha G_1(x, t)| \leq C_N t^{-(n+|\alpha|+2h)/2} B_N(|x|, t).$$

**[22] Proposition 4.2.** *For fixed  $\varepsilon$  and  $R$ , there exist positive numbers  $m$ ,  $C$  and  $N > n$  such that*

$$|\partial_t^h D_x^\alpha G_2(x, t)| \leq C e^{-\frac{t}{2m}} B_N(|x|, t).$$

Next we will come to consider  $G_3(x, t)$ . Now we list some lemmas which are useful in dealing with the higher frequency part.

[23] **Lemma 4.3.** *If  $\text{supp } \hat{f}(\xi) \subset O_R := \{\xi; |\xi| > R\}$ , and  $\hat{f}(\xi)$  satisfies*

$$|\hat{f}(\xi)| \leq C, \quad |D_\xi^\beta \hat{f}(\xi)| \leq C|\xi|^{-1-|\beta|}, \quad |\beta| \geq 1,$$

*then there exist distributions  $f_1(x)$ ,  $f_2(x)$ , and constant  $C_0$  such that*

$$f(x) = f_1(x) + f_2(x) + C_0\delta(x),$$

*where  $\delta(x)$  is the Dirac function. Furthermore, for positive integer  $2N > n + |\alpha|$ ,*

$$|D_x^\alpha f_1(x)| \leq C(1 + |x|^2)^{-N},$$

$$\|f_2\|_{L^1} \leq C, \quad \text{supp } f_2(x) \subset \{x; |x| < 2\varepsilon_1\},$$

*with  $\varepsilon_1$  being sufficiently small.*

[24] **Lemma 4.4.** *For any  $N > 0$ ,  $\tau \geq 0$ , we have that*

$$\begin{aligned} \int_{|z|=1} \left(1 + \frac{|x + tz|^2}{1 + \tau}\right)^{-N} dS_z &\leq C(1 + t)^{2N} \left(1 + \frac{|x|^2}{1 + \tau}\right)^{-N}. \\ \int_{|z| \leq 1} \frac{\left(1 + \frac{|x + tz|^2}{1 + \tau}\right)^{-N}}{\sqrt{1 + \tau}} dV_z &\leq C(1 + t)^{2N} \left(1 + \frac{|x|^2}{1 + \tau}\right)^{-N}. \end{aligned}$$

The proof of Lemma 4.3 and Lemma 4.4 can be seen in [24].

The following Kirchhoff formulas can be seen in [2, 4].

[25] **Lemma 4.5.** *Assume that  $w(x, t)$  is the fundamental solution of the following wave equation with  $c = 1$ ,*

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, \\ w|_{t=0} = 0, \\ \partial_t w|_{t=0} = \delta(x). \end{cases}$$

*There are constants  $a_\alpha$ ,  $b_\alpha$  depending only on the spatial dimension  $n \geq 1$  such that, if  $h \in C^\infty(\mathbb{R}^n)$ , then*

$$(w * h)(x, t) = \sum_{0 \leq |\alpha| \leq \frac{n-3}{2}} a_\alpha t^{|\alpha|+1} \int_{|z|=1} D^\alpha h(x + tz) z^\alpha dS_z,$$

$$(w_t * h)(x, t) = \sum_{0 \leq |\alpha| \leq \frac{n-1}{2}} b_\alpha t^{|\alpha|} \int_{|z|=1} D^\alpha h(x + tz) z^\alpha dS_z,$$

for odd  $n$ , and

$$(w * h)(x, t) = \sum_{0 \leq |\alpha| \leq \frac{n-2}{2}} a_\alpha t^{|\alpha|+1} \int_{|z| \leq 1} \frac{D^\alpha h(x + tz) z^\alpha}{\sqrt{1 - |z|^2}} dz,$$

$$(w_t * h)(x, t) = \sum_{0 \leq |\alpha| \leq \frac{n}{2}} b_\alpha t^{|\alpha|} \int_{|z| \leq 1} \frac{D^\alpha h(x + tz) z^\alpha}{\sqrt{1 - |z|^2}} dz,$$

for even  $n$ . Here  $dS_z$  denotes surface measure on the unit sphere in  $\mathbb{R}^n$ .

By denoting  $\lambda_\eta = \sqrt{\eta - 4}$  and then taking the Taylor expansion for  $\lambda_\eta$  in  $\eta$ , we have that

$$\lambda_\eta = 2\sqrt{-1} + \sum_{j=1}^{m-1} a_j \eta^j + O(\eta^m).$$

Since

$$\mu_\pm(\xi) = \frac{-1 \pm \sqrt{1 - 4|\xi|^2}}{2} = \frac{1}{2}(-1 \pm |\xi| \sqrt{|\xi|^{-2} - 4}),$$

we have that, when  $\xi$  is sufficiently large,

$$\mu_\pm(\xi) = \frac{1}{2}(-1 \pm 2\sqrt{-1}|\xi| \pm (\sum_{j=1}^{m-1} a_j |\xi|^{1-2j})) + O(|\xi|^{1-2m}).$$

$$\mu_0^{-1}(\xi) = \frac{1}{\sqrt{1 - 4|\xi|^2}} = |\xi|^{-1}(-\frac{\sqrt{-1}}{2} + O(|\xi|^{-2})).$$

This implies that

$$\begin{aligned} e^{\mu_\pm(\xi)t} &= e^{-t/2} e^{\pm\sqrt{-1}|\xi|t} (1 + (\sum_{j=1}^{m-1} (\pm a_j) |\xi|^{1-2j})t + \dots \\ &\quad + \frac{1}{m!} (\sum_{j=1}^{m-1} (\pm a_j) |\xi|^{1-2j})^m t^m + R^\pm(\xi, t)), \end{aligned}$$

where  $R^\pm(\xi, t) \leq (1 + t)^{m+1} (1 + |\xi|)^{1-2m}$ .

Denote

$$\hat{w}(\xi, t) = (2\pi)^{-n/2} \sin(|\xi|t)/|\xi|, \quad \hat{w}_t = (2\pi)^{-n/2} \cos(|\xi|t).$$

Since

$$\partial_t^h \hat{G}^+(\xi, t) = \frac{(\mu_+(\xi))^h}{\mu_0} e^{\mu_+(\xi)t}, \quad \partial_t^h \hat{G}^-(\xi, t) = -\frac{(\mu_-(\xi))^h}{\mu_0} e^{\mu_-(\xi)t}.$$

By a direct and a little tedious calculation we get that,

$$\begin{aligned}\partial_t^h \hat{G}_3(\xi, t) &= e^{-t/2} \hat{w}_t \left( \sum_{j=0}^{h-1} p_{1j}^1(t) q_{1j}^1(\xi) + \sum_{j=1}^{2m-2} p_{2j}^1(t) q_{2j}^1(\xi) + \hat{R}^1(\xi, t) \right) \\ &\quad + e^{-t/2} \hat{w} \left( \sum_{j=0}^h p_{1j}^2(t) q_{1j}^2(\xi) + \sum_{j=1}^{2m-2} p_{2j}^2(t) q_{2j}^2(\xi) + \hat{R}^2(\xi, t) \right),\end{aligned}$$

here

$$p_{1j}^1(t) \leq C(1+t)^{h-1-j}, \quad q_{1j}^1(\xi) = \chi_3(\xi) |\xi|^j, \quad 0 \leq j \leq h-1;$$

$$p_{2j}^1(t) \leq C(1+t)^{h+j}, \quad q_{2j}^1(\xi) = \chi_3(\xi) |\xi|^{-j}, \quad 1 \leq j \leq 2m-2;$$

$$p_{1j}^2(t) \leq C(1+t)^{h-j}, \quad q_{1j}^2(\xi) = \chi_3(\xi) |\xi|^j, \quad 0 \leq j \leq h;$$

$$p_{2j}^2(t) \leq C(1+t)^{h+j}, \quad q_{2j}^2(\xi) = \chi_3(\xi) |\xi|^{-j}, \quad 1 \leq j \leq 2m-2;$$

$$|\hat{R}^1(\xi, t)|, \quad |\hat{R}^2(\xi, t)| \leq C(1+t)^{m+1} (1+|\xi|)^{h+1-2m}.$$

In the following we denote  $q_{1j}^1(D_x)$  ( $0 \leq j \leq h-1$ ),  $q_{1j}^2(D_x)$  ( $0 \leq j \leq h$ ),  $q_{2j}^1(D_x)$  ( $0 \leq j \leq 2m-2$ ),  $q_{2j}^2(D_x)$  ( $0 \leq j \leq 2m-2$ ),  $w(D_x, t)$ ,  $w_t(D_x, t)$ ,  $R^1(D_x, t)$ ,  $R^2(D_x, t)$  the pseudo-differential operators with symbols  $q_{1j}^1(\xi)$  ( $0 \leq j \leq h-1$ ),  $q_{1j}^2(\xi)$  ( $0 \leq j \leq h$ ),  $q_{2j}^1(\xi)$  ( $0 \leq j \leq 2m-2$ ),  $q_{2j}^2(\xi)$  ( $0 \leq j \leq 2m-2$ ),  $\hat{w}(\xi, t)$ ,  $\hat{w}_t(\xi, t)$ ,  $\hat{R}^1(\xi, t)$ ,  $\hat{R}^2(\xi, t)$  respectively. It is easy to get that, for any multi-indexes  $\beta$ ,  $|\beta| \geq 1$ ,

$$|D_\xi^\beta q_{1j}^1(\xi)| \leq C|\xi|^{-1-|\beta|}, \quad |D_\xi^\beta q_{2j}^2(\xi)| \leq C|\xi|^{-1-|\beta|}, \quad 1 \leq j \leq 2m-2;$$

$$|D_\xi^\beta \chi_3(\xi)| \leq C|\xi|^{-1-|\beta|}, \quad |D_\xi^\beta (|\xi|^{-1} \chi_3(\xi))| \leq C|\xi|^{-1-|\beta|};$$

$$\text{supp } q_{1j}^1(0 \leq j \leq h-1), \text{ supp } q_{1j}^2(0 \leq j \leq h) \subset O_{R-1} = \{\xi; |\xi| > R-1\};$$

$$\text{supp } q_{2j}^1(1 \leq j \leq 2m-2), \text{ supp } q_{2j}^2(1 \leq j \leq 2m-2) \subset O_{R-1}.$$

By Lemma <sup>23</sup>4.3 we have the following lemma.

**[28] Lemma 4.6.** *For  $R$  being sufficiently large, there exist distributions  $\bar{q}_{1j}^1(x)$ ,  $\bar{q}_{1j}^2(x)$ ,  $0 \leq j \leq h-1$ ;  $\bar{q}_{1j}^2(x)$ ,  $\bar{q}_{1j}^2(x)$ ,  $0 \leq j \leq h$ ;  $\bar{q}_{2j}^1(x)$ ,  $\bar{q}_{2j}^1(x)$ ,  $1 \leq j \leq$*

$2m - 2; \bar{q}_{2j}^2(x), \tilde{q}_{2j}^2(x), 1 \leq j \leq 2m - 2$  and constant  $C_0$  such that

$$q_{1j}^1(D_x)\delta(x) = (-\Delta)^{\frac{j}{2}}(\bar{q}_{1j}^1 + \tilde{q}_{1j}^1 + C_0\delta(x)), \quad 0 \leq j \leq h - 1;$$

$$q_{1j}^2(D_x)\delta(x) = (-\Delta)^{\frac{j}{2}}(\bar{q}_{1j}^2 + \tilde{q}_{1j}^2 + C_0\delta(x)), \quad 0 \leq j \leq h;$$

$$q_{2j}^1(D_x)\delta(x) = \bar{q}_{2j}^1 + \tilde{q}_{2j}^1 + C_0\delta(x), \quad 1 \leq j \leq 2m - 2;$$

$$q_{2j}^2(D_x)\delta(x) = \bar{q}_{2j}^2 + \tilde{q}_{2j}^2 + C_0\delta(x), \quad 1 \leq j \leq 2m - 2;$$

and

$$|D_x^\alpha \bar{q}_{1j}^1|(0 \leq j \leq h - 1), \quad |D_x^\alpha \bar{q}_{1j}^2|(0 \leq j \leq h) \leq C(1 + |x|^2)^{-N};$$

$$|D_x^\alpha \bar{q}_{2j}^1|(1 \leq j \leq 2m - 2), \quad |D_x^\alpha \bar{q}_{2j}^2|(1 \leq j \leq 2m - 2) \leq C(1 + |x|^2)^{-N};$$

$$\|\tilde{q}_{1j}^1\|_{L^1}(0 \leq j \leq h - 1), \quad \|\tilde{q}_{1j}^2\|_{L^1}(0 \leq j \leq h) \leq C;$$

$$\|\tilde{q}_{2j}^1\|_{L^1}(1 \leq j \leq 2m - 2), \quad \|\tilde{q}_{2j}^2\|_{L^1}(1 \leq j \leq 2m - 2) \leq C;$$

$$\text{supp } \tilde{q}_{1j}^1(0 \leq j \leq h - 1), \quad \text{supp } \tilde{q}_{1j}^2(0 \leq j \leq h) \subset \{x; |x| < 2\varepsilon_1\};$$

$$\text{supp } \tilde{q}_{2j}^1(1 \leq j \leq 2m - 2), \quad \text{supp } \tilde{q}_{2j}^2(1 \leq j \leq 2m - 2) \subset \{x; |x| < 2\varepsilon_1\},$$

with  $\varepsilon_1$  being sufficiently small.

Let

$$Q_{1j}^1(x) = \tilde{q}_{1j}^1(x) + C_0\delta(x), \quad 0 \leq j \leq h - 1;$$

$$Q_{1j}^2(x) = \tilde{q}_{1j}^2(x) + C_0\delta(x), \quad 0 \leq j \leq h;$$

$$Q_{2j}^1(x) = \tilde{q}_{2j}^1(x) + C_0\delta(x), \quad 1 \leq j \leq 2m - 2;$$

$$Q_{2j}^2(x) = \tilde{q}_{2j}^2(x) + C_0\delta(x), \quad 1 \leq j \leq 2m - 2,$$

and

$$L_{1j}^1(x, t) = p_{1j}^1(t)w_t(D_x, t)(-\Delta)^{\frac{j}{2}}Q_{1j}^1(x), \quad 0 \leq j \leq h - 1;$$

$$L_{1j}^2(x, t) = p_{1j}^2(t)w(D_x, t)(-\Delta)^{\frac{j}{2}}Q_{1j}^2(x), \quad 0 \leq j \leq h;$$

$$L_{2j}(x, t) = p_{2j}^1(t)w_t(D_x, t)Q_{2j}^1(x) + p_{2j}^2(t)w(D_x, t)Q_{2j}^2(x), \quad 1 \leq j \leq 2m - 2,$$

we have the following proposition.

**Proposition 4.7.** *For  $R$  sufficiently large, there exists distribution*

$$K_m^h(x, t) = e^{-t/2} \left( \sum_{j=0}^{h-1} L_{1j}^1(x, t) + \sum_{j=0}^h L_{1j}^2(x, t) + \sum_{j=1}^{2m-2} L_{2j}(x, t) \right)$$

such that for  $m \geq \lceil \frac{|\alpha|+n+h+3}{2} \rceil$ , we have that

$$|D_x^\alpha(\partial_t^h G_3 - K_m^h)(x, t)| \leq C e^{-t/4} B_N(|x|, t).$$

The proof of Proposition 4.7 can be seen in [11].

By Proposition 4.1, Proposition 4.2 and Proposition 4.7, we have the following proposition on the Green function.

**Proposition 4.8.** *For any integer  $h \geq 0$ , any multi-index  $\alpha$ , and  $m \geq \lceil \frac{|\alpha|+n+h+3}{2} \rceil$ , we have that*

$$|D_x^\alpha(\partial_t^h G - K_m^h)(x, t)| \leq C(1+t)^{-(n+|\alpha|+2h)/2} B_N(|x|, t),$$

where  $N > n$  can be big enough.

## 5 Pointwise estimates

In this section, we aim at verifying that the solution obtained in Theorem 2.1 satisfies the pointwise decay estimates expressed in Theorem 2.4.

By Duhamel's principle, the solution to (1.1)(1.2) can be expressed as following,

$$\begin{aligned} u(x, t) = & G(x - \cdot, t) * (u_0 + u_1)(\cdot) + \partial_t G(x - \cdot, t) * u_0(\cdot) \\ & - \int_0^t G(x - \cdot, t - \tau) * (|u|^\theta u)(\cdot, \tau) d\tau. \end{aligned}$$

We denote the solution to the corresponding linear dissipative wave equation as  $\bar{u}$ , then

$$\bar{u}(x, t) := G(x - \cdot, t) * (u_0 + u_1)(\cdot) + \partial_t G(x - \cdot, t) * u_0(\cdot).$$

Denote

$$\tilde{u}(x, t) := \int_0^t G(x - \cdot, t - \tau) * (|u|^\theta u)(\cdot, \tau) d\tau,$$

then the solution  $u$  to (1.1) can be expressed as:  $u = \bar{u} - \tilde{u}$ .

In [11], the following pointwise estimate of the solution  $\bar{u}$  to the linear problem is obtained.

**1r** **Theorem 5.1.** <sup>[LW]</sup><sub>[7]</sub> Assume that  $(u_0, u_1) \in H^{s+1} \times H^s$ ,  $s > n$  is an integer, and for any multi-index  $\alpha \in \mathbb{Z}^n$ ,  $|\alpha| < s - \frac{n}{2}$ , there exists  $r > \frac{n}{2}$  such that  $|D_x^\alpha u_0(x)| + |D_x^\alpha u_1(x)| \leq C(1 + |x|^2)^{-r}$ , then the solution  $\bar{u}$  satisfies, for  $|\alpha| + h < s - n$ ,

$$|\partial_t^h D_x^\alpha \bar{u}(x, t)| \leq C(1 + t)^{-(n+|\alpha|+2h)/2} (1 + \frac{|x|^2}{1+t})^{-r}. \quad (5.1) \quad \text{1r1}$$

Now we give some lemmas which will be used later.

**41** **Lemma 5.2.** Assume  $n \geq 1$ , then the following inequalities hold,  
(1). If  $\tau \in [0, t]$ , and  $A^2 \geq t$ , then

$$(1 + \frac{A^2}{1+\tau})^{-n} \leq 2^n (\frac{1+\tau}{1+t})^n (1 + \frac{A^2}{1+t})^{-n}.$$

(2). If  $A^2 \leq t$ , then  $1 \leq 2^n (1 + \frac{A^2}{1+t})^{-n}$ .

**42** **Lemma 5.3.** Assume that  $0 \leq \tau \leq t$  and  $h(x, \tau)$  satisfies

$$D_x^\alpha h(x, \tau) \leq C(1 + \tau)^{-\frac{\theta n + |\alpha|}{2}} (1 + \frac{|x|^2}{1+\tau})^{-r},$$

then we have that,

$$(1). \quad \int_{|z|=1} |D_x^\alpha h(x + tz, \tau)| dS_z \leq C(1 + \tau)^{-\frac{\theta n + |\alpha|}{2}} (1 + t)^{2r} (1 + \frac{|x|^2}{1+\tau})^{-r}.$$

$$(2). \quad \int_{|z| \leq 1} \frac{|D_x^\alpha h(x + tz, \tau)|}{\sqrt{1-|z|^2}} dV_z \leq C(1 + \tau)^{-\frac{\theta n + |\alpha| - 1}{2}} (1 + t)^{2r} (1 + \frac{|x|^2}{1+\tau})^{-r}.$$

*Proof.* (1). By using Lemma <sup>24</sup><sub>(4.4)</sub><sub>1</sub>,

$$\begin{aligned} \int_{|z|=1} |D_x^\alpha h(x + tz, \tau)| dS_z &\leq C \int_0^t (1 + \tau)^{-\frac{\theta n + |\alpha|}{2}} (1 + \frac{|x+tz|^2}{1+\tau})^{-r} dS_z \\ &\leq C(1 + \tau)^{-\frac{\theta n + |\alpha|}{2}} (1 + t)^{2r} (1 + \frac{|x|^2}{1+\tau})^{-r}. \end{aligned}$$

(2). By Hölder inequality and Lemma <sup>24</sup><sub>(4.4)</sub><sub>2</sub>,

$$\begin{aligned} &\int_{|z| \leq 1} \frac{|D_x^\alpha h(x + tz, \tau)|}{\sqrt{1-|z|^2}} dV_z \\ &\leq (\int_{|z| \leq 1} |D_x^\alpha h(x + tz, \tau)|^3 dV_z)^{\frac{1}{3}} (\int_{|z| \leq 1} (\frac{1}{\sqrt{1-|z|^2}})^{\frac{3}{2}} dV_z)^{\frac{2}{3}} \\ &\leq C(1 + \tau)^{-\frac{\theta n + |\alpha| - 1}{2}} (\int_{|z| \leq 1} \frac{(1 + \frac{|x+tz|^2}{1+\tau})^{-3r}}{\sqrt{1+\tau}} dV_z)^{\frac{1}{3}} (\int_0^1 (1 - r^2)^{-\frac{3}{4}} r^{n-1} dr)^{\frac{2}{3}} \\ &\leq C(1 + \tau)^{-\frac{\theta n + |\alpha| - 1}{2}} (1 + t)^{2r} (1 + \frac{|x|^2}{1+\tau})^{-r}. \end{aligned}$$

Thus we complete the proof of Lemma <sup>42</sup><sub>5.3</sub>. □

*Proof of Theorem 2.4.* For  $s > n$ , denote

$$\varphi_\alpha(x, t) := (1 + t)^{\frac{n+|\alpha|}{2}} (B_r(|x|, t))^{-1}, \quad r > \frac{n}{2},$$

$$M(T) := \sup_{\substack{(x, \tau) \in \mathbb{R}^n \times [0, T) \\ |\alpha| + h < s - n}} |D_x^\alpha \partial_\tau^h u(x, \tau)| \varphi_\alpha(x, \tau).$$

Now we come to make estimates to  $\tilde{u}(x, t)$  under the assumption that  $s > n$  and  $\theta \geq 2 + \lfloor \frac{1}{n} \rfloor$ .

By induction argument, we obtain the following expression,

$$\begin{aligned} \partial_t^h \tilde{u}(x, t) &= \partial_t^h \int_0^t G(t - \tau) * (|u|^\theta u)(\tau) d\tau \\ &= \sum_{j=0}^{(h-1)_+} \partial_t^j G(t) * \partial_t^{(h-1)_+ - j} (|u|^\theta u)(0) + \int_0^t G(t - \tau) * \partial_\tau^h (|u|^\theta u)(\tau) d\tau \\ &=: J_1 + J_2, \end{aligned} \tag{5.2} \quad \boxed{J}$$

where  $(h-1)_+ = \max\{h-1, 0\}$ .

From (3.10) and (3.11), we know that  $\partial_t^{(h-1)_+ - j} (|u|^\theta u)(x, 0)$  is a polynomial with arguments  $\Delta^i u_0(x)$  and  $\Delta^k u_1(x)$ ,  $0 \leq i \leq \lfloor \frac{(h-1)_+ - j}{2} \rfloor$ ,  $0 \leq k \leq \lfloor \frac{(h-1)_+ - j - 1}{2} \rfloor$ . By using the similar estimates as that for  $\bar{u}$  in [11], we obtain the following estimate for  $D_x^\alpha J_1$ ,

$$|D_x^\alpha J_1| \leq C E_0 (1 + t)^{-\frac{n+|\alpha|}{2}} B_N(|x|, t).$$

As for the estimate to  $D_x^\alpha J_2$ , we divide it as following ,

$$\begin{aligned} D_x^\alpha J_2 &= \int_0^{\frac{t}{2}} \int_{\{y: |x| \leq 2|y|\}} D_x^\alpha (G - K_m^0)(x - y, t - \tau) \partial_\tau^h (|u|^\theta u)(y, \tau) dy d\tau \\ &\quad + \int_0^{\frac{t}{2}} \int_{\{y: |x| \geq 2|y|\}} D_x^\alpha (G - K_m^0)(x - y, t - \tau) \partial_\tau^h (|u|^\theta u)(y, \tau) dy d\tau \\ &\quad + \int_{\frac{t}{2}}^t \int_{\{y: |x| \leq 2|y|\}} (G - K_m^0)(x - y, t - \tau) D_y^\alpha \partial_\tau^h (|u|^\theta u)(y, \tau) dy d\tau \\ &\quad + \int_{\frac{t}{2}}^t \int_{\{y: |x| \geq 2|y|\}} (G - K_m^0)(x - y, t - \tau) D_y^\alpha \partial_\tau^h (|u|^\theta u)(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} K_m^0(x - y, t - \tau) D_y^\alpha \partial_\tau^h (|u|^\theta u)(y, \tau) dy d\tau \\ &=: J_{21} + J_{22} + J_{23} + J_{24} + J_{25}. \end{aligned}$$

Next we estimate  $J_{2i}$  ( $i = 1, 2, 3, 4, 5$ ) respectively by using Proposition 4.8 (with  $N \geq r$ ) and the fact that  $B(|x|, t) \leq 1$  and is an increasing function of  $t$  and decreasing function of  $|x|$ .



By the definition of  $M(T)$ , we have that

$$|J_{21}| \leq C \int_0^{\frac{t}{2}} \int_{\{y: |x| \leq 2|y|\}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} B_N(|x-y|, t-\tau) M(T)^{\theta+1} (1+\tau)^{-\frac{(\theta+1)n+2h}{2}} B_r(|y|, \tau) dy d\tau.$$

Now we estimate  $J_{21}$  in two cases.

Case 1.  $|x|^2 \geq t$ . We have

$$\begin{aligned} |J_{21}| &\leq CM(T)^{\theta+1} \int_0^{\frac{t}{2}} \int_{\{y: |x| \leq 2|y|\}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} \\ &\quad B_N(|x-y|, t-\tau) (1+\tau)^{-\frac{(\theta+1)n}{2}} B_r(|x|, \tau) dy d\tau \\ &\leq CM(T)^{\theta+1} B_r(|x|, t) \int_0^{\frac{t}{2}} \int_{\{y: |x| \leq 2|y|\}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} \\ &\quad B_N(|x-y|, t-\tau) (1+\tau)^{-\frac{(\theta+1)n}{2}} \left(\frac{1+\tau}{1+t}\right)^r dy d\tau \\ &\leq CM(T)^{\theta+1} B_r(|x|, t) \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{|\alpha|}{2}} (1+\tau)^{-\frac{(\theta+1)n}{2}} \left(\frac{1+\tau}{1+t}\right)^{\frac{n}{2}} d\tau \\ &\leq CM(T)^{\theta+1} B_r(|x|, t) (1+t)^{-\frac{n+|\alpha|}{2}}, \end{aligned}$$

here in the second inequality we used Lemma ~~5.2~~<sup>4.1</sup> (1).

Case 2.  $|x|^2 \leq t$ . We have

$$\begin{aligned} |J_{21}| &\leq CM(T)^{\theta+1} \int_0^{\frac{t}{2}} \int_{\{y: |x| \leq 2|y|\}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} \\ &\quad (1+\tau)^{-\frac{(\theta+1)n+2h}{2}} B_r(|y|, \tau) dy d\tau \\ &\leq CM(T)^{\theta+1} \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} (1+\tau)^{-\frac{\theta n+2h}{2}} d\tau \\ &\leq CM(T)^{\theta+1} (1+t)^{-\frac{n+|\alpha|}{2}} \\ &\leq CM(T)^{\theta+1} B_r(|x|, t) (1+t)^{-\frac{n+|\alpha|}{2}}, \end{aligned}$$

here in the last inequality we used Lemma ~~5.2~~<sup>4.1</sup> (2).

Combining the two cases, we have that

$$|J_{21}| \leq CM(T)^{\theta+1} (\varphi_\alpha(x, t))^{-1}.$$

For  $J_{22}$ , we have

$$|J_{22}| \leq C \int_0^{\frac{t}{2}} \int_{\{y: |x| \geq 2|y|\}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} B_N(|x-y|, t-\tau) M(T)^{\theta+1} (1+\tau)^{-\frac{(\theta+1)n+2h}{2}} B_r(|y|, \tau) dy d\tau,$$

noticing that if  $|x| \geq 2|y|$ , then  $|x-y| \geq \frac{|x|}{2}$ , it yields that

$$\begin{aligned} |J_{22}| &\leq CM(T)^{\theta+1} \int_0^{\frac{t}{2}} \int_{\{y: |x| \geq 2|y|\}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} \\ &\quad B_N(|x|, t) (1+\tau)^{-\frac{(\theta+1)n+2h}{2}} B_r(|y|, \tau) dy d\tau \\ &\leq CM(T)^{\theta+1} B_N(|x|, t) \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n+|\alpha|}{2}} (1+\tau)^{-\frac{\theta n+2h}{2}} d\tau \\ &\leq CM(T)^{\theta+1} B_N(|x|, t) (1+t)^{-\frac{n+|\alpha|}{2}} \\ &\leq CM(T)^{\theta+1} (\varphi_\alpha(x, t))^{-1}. \end{aligned}$$

For  $J_{23}$ , by using the monotonic properties of  $B(|x|, t)$  with respect to  $|x|$  and  $t$ , we have that

$$\begin{aligned}
|J_{23}| &\leq C \int_{\frac{t}{2}}^t \int_{\{y: |x| \leq 2|y|\}} (1+t-\tau)^{-\frac{n}{2}} B_N(|x-y|, t-\tau) \\
&\quad M(T)^{\theta+1} (1+\tau)^{-\frac{(\theta+1)n+|\alpha|+2h}{2}} B_r(|y|, \tau) dy d\tau \\
&\leq CM(T)^{\theta+1} \int_{\frac{t}{2}}^t \int_{\{y: |x| \leq 2|y|\}} (1+t-\tau)^{-\frac{n}{2}} \\
&\quad B_N(|x-y|, t-\tau) (1+\tau)^{-\frac{(\theta+1)n+|\alpha|+2h}{2}} B_r(|x|, t) dy d\tau \\
&\leq CM(T)^{\theta+1} B_r(|x|, t) \int_{\frac{t}{2}}^t (1+\tau)^{-\frac{(\theta+1)n+|\alpha|+2h}{2}} d\tau \\
&\leq CM(T)^{\theta+1} B_r(|x|, t) (1+t)^{-\frac{n+|\alpha|}{2}} \\
&\leq CM(T)^{\theta+1} (\varphi_{\alpha, h}(x, t))^{-1}.
\end{aligned}$$

For  $J_{24}$ , similar to  $J_{22}$  we have that

$$\begin{aligned}
|J_{24}| &\leq C \int_{\frac{t}{2}}^t \int_{\{y: |x| \geq 2|y|\}} (1+t-\tau)^{-\frac{n}{2}} B_N(|x-y|, t-\tau) \\
&\quad M(T)^{\theta+1} (1+\tau)^{-\frac{(\theta+1)n+|\alpha|+2h}{2}} B_r(|y|, \tau) dy d\tau \\
&\leq CM(T)^{\theta+1} \int_{\frac{t}{2}}^t \int_{\{y: |x| \geq 2|y|\}} (1+t-\tau)^{-\frac{n}{2}} \\
&\quad B_N(|x|, t) (1+\tau)^{-\frac{(\theta+1)n+|\alpha|+2h}{2}} B_r(|y|, \tau) dy d\tau \\
&\leq CM(T)^{\theta+1} B_N(|x|, t) \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{n}{2}} (1+\tau)^{-\frac{\theta n+|\alpha|+2h}{2}} d\tau \\
&\leq CM(T)^{\theta+1} B_N(|x|, t) (1+t)^{-\frac{n+|\alpha|+2h}{2}} \\
&\leq CM(T)^{\theta+1} (\varphi_{\alpha}(x, t))^{-1}.
\end{aligned}$$

Finally we come to estimate  $J_{25}$ . From the definition of  $K_m^0$  we have

$$\begin{aligned}
|J_{25}| &= \left| \int_0^t \int_{\mathbb{R}^n} e^{-\frac{t-\tau}{2}} p_{10}^2(t-\tau) w_t(D_x, t-\tau) (\tilde{q}_{10}^2 + C_0 \delta)(x-y) \right. \\
&\quad + \sum_{j=1}^{2m-2} [p_{2j}^1(t-\tau) w_t(D_x, t-\tau) (\tilde{q}_{2j}^1 + C_0 \delta)(x-y) \\
&\quad \left. + p_{2j}^2(t-\tau) w(D_x, t-\tau) (\tilde{q}_{2j}^2 + C_0 \delta)(x-y)] \right\} D_y^\alpha \partial_\tau^h (|u|^\theta u)(y, \tau) dy d\tau \\
&\leq \left| \int_0^t \int_{\mathbb{R}^n} e^{-\frac{t-\tau}{2}} p_{10}^2(t-\tau) w(D_x, t-\tau) \tilde{q}_{10}^2(x-y) \right. \\
&\quad + \sum_{j=1}^{2m-2} [p_{2j}^1(t-\tau) w_t(D_x, t-\tau) \tilde{q}_{2j}^1(x-y) \\
&\quad \left. + p_{2j}^2(t-\tau) w(D_x, t-\tau) \tilde{q}_{2j}^2(x-y)] \right\} D_y^\alpha (|u|^\theta u)(y, \tau) dy d\tau \\
&\quad + \left| \int_{\frac{t}{2}}^t \int_{\mathbb{R}^n} e^{-\frac{t-\tau}{2}} \left\{ \sum_{j=1}^{h-1} p_{10}^2(t-\tau) w(D_x, t-\tau) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{2m-2} [p_{2j}^1(t-\tau) w_t(D_x, t-\tau) + p_{2j}^2(t-\tau) w(D_x, t-\tau)] \right\} \right. \\
&\quad \left. C_0 \delta(x-y) D_y^\alpha (|u|^\theta u)(y, \tau) dy d\tau \right| \\
&=: K_1 + K_2.
\end{aligned}$$

(5.3) j25

By using Lemma [25](#), Lemma [28](#), Lemma [42](#) and the fact that  $(1 + \frac{|y|^2}{1+t})^{-1} \leq C(1 + \frac{|x|^2}{1+t})^{-1}$ , if  $|x - y| \leq 2\varepsilon_1$ ,  $K_1$  can be estimated as follows,

$$\begin{aligned} K_1 &\leq C \int_0^t \int_{\mathbb{R}^n} e^{-\frac{t-\tau}{4}} (|\tilde{q}_{10}^2| + \sum_{j=1}^{2m-2} (|\tilde{q}_{2j}^1| + |\tilde{q}_{2j}^2|))(x - y) \\ &\quad M(T)^{\theta+1} (1 + \tau)^{-\frac{(\theta+1)n+|\alpha|}{2}} (1 + \frac{|y|^2}{1+\tau})^{-r} dy d\tau \\ &\leq CM(T)^{\theta+1} (1 + t)^{-\frac{n+|\alpha|}{2}} (1 + \frac{|x|^2}{1+t})^{-r}. \end{aligned} \quad (5.4) \quad \boxed{\text{k1}}$$

By using Lemma [25](#) and Lemma [42](#),  $K_2$  can be controlled by

$$\begin{aligned} K_2 &\leq C \int_0^t e^{-\frac{t-\tau}{4}} M(T)^{\theta+1} (1 + \tau)^{-\frac{(\theta+1)n+|\alpha|}{2}} (1 + \frac{|x|^2}{1+\tau})^{-r} d\tau \\ &\leq CM(T)^{\theta+1} (1 + t)^{-\frac{n+|\alpha|}{2}} (1 + \frac{|x|^2}{1+t})^{-r}. \end{aligned} \quad (5.5) \quad \boxed{\text{k2}}$$

[\(5.3\)](#), [\(5.4\)](#) and [\(5.5\)](#) yield that

$$|J_{25}| \leq CM(T)^{\theta+1} (\varphi_\alpha(x, t))^{-1}.$$

Combining the estimates for  $J_{2i}, i = 1, 2, 3, 4, 5$ , we have that

$$|D_x^\alpha \partial_t^h \tilde{u}(x, t)| \leq CM(T)^{\theta+1} (\varphi_\alpha(x, t))^{-1}. \quad (5.6) \quad \boxed{\text{pe2}}$$

[\(5.6\)](#) combined with [\(5.1\)](#) Theorem [1r](#) yields that,

$$M(T) \leq C(E_0 + M(T)^{\theta+1}).$$

Since  $\theta \geq 2$ , we have that  $M(T) \leq CE_0$  if  $E_0$  is suitably small. It yields that

$$|D_x^\alpha \partial_t^h u(x, t)| \leq CE_0 (1 + t)^{-\frac{n+|\alpha|}{2}} B_r(|x|, t).$$

Thus Theorem [2.4](#) is proved.  $\square$

## Acknowledgement

This work was partially supported by Grant-in-Aid for JSPS Fellows. Also, the author would give thanks to professor Weike Wang for the helpful discussion.

## References

- BP [1] V. Belleri, V. Pata, *Attractors for semilinear strongly damped wave equations on  $\mathbb{R}^3$* , Disc. and Cont. Dynamical Systems **7** (2001), 719-735.

- [E] [2] L. C. Evans, *"Partial Differential Equations," Graduate Studies in Math., 19*, Amer. Math. Soc., Providence, RI, 1998.
- [HO] [3] T. Hosono, T. Ogawa, *Large time behavior and  $L^p - L^q$  estimate of solutions of 2-dimensional nonlinear damped wave equations*, J. Differential Equations **203** (2004), 82-118.
- [HZ] [4] D. Hoff, K. Zumbrun, *Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves*, Z. angew Math. Phys. **48** (1997), 1-18.
- [Ik] [5] R. Ikehata, *A remark on a critical exponent for the semilinear dissipative wave equation in the one dimensional half space*, Differential Integral Equations **16** (2003), 727-736.
- [IMN] [6] R. Ikehata, Y. Miyaoka, T. Nakatake, *Decay estimate of solution for dissipative wave equation in  $\mathbb{R}^n$  with lower power nonlinearities*, J. Math. Soc. Japan **56(2)** (2004), 365-373.
- [INZ] [7] R. Ikehata, K. Nishihara, H. Zhao, *Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption*, J. Differential Equations **226(1)** (2006), 1-29.
- [IO] [8] R. Ikehata, M. Ohta, *Critical exponent for semilinear dissipative wave equation in  $\mathbb{R}^n$* , J. Math. Anal. Appl. **269(1)** (2002), 87-97.
- [KNO] [9] S. Kawashima, M. Nakao, K. Ono, *on the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term*, J. Math. Soc. Japan **47** (1995), 617-653.
- [KS] [10] N. I. Karachaios, N. M. Stavrakakis, *Estimates on the dimension of a global attractor for a semilinear dissipative wave equation on  $\mathbb{R}^n$* , Disc. and Cont. Dynamical Systems **8** (2002), 939-951.
- [LW] [11] Y. Liu, W. Wang, *The pointwise estimates of solutions for dissipative wave equation in multi-dimensions*, Discrete and Continuous Dynamical Systems, **20** (2008), 1013-1028.
- [LZ] [12] T.-T. Li, Y. Zhou, *Breakdown of solutions to  $\square u + u_t = |u|^{1+\alpha}$* , Discrete Contin. Dyn. Syst. **1** (1995), no. 4, 503-520.
- [Na] [13] T. Narazaki,  *$L^p - L^q$  estimates for damped wave equations and their applications to semi-linear problem*, J. Math. Soc. Japan **56** (2004), 585-626.

- [Ni] [14] K. Nishihara,  *$L^p - L^q$  estimates of solutions to the damped wave equation in 3-dimensional space and their application*, Math. Z. **244** (2003), 631-649.
- [Ni0] [15] K. Nishihara, *Global asymptotics for the damped wave equation with absorption in higher dimensional space*, J. Math. Soc. Japan **58**( 3) (2006), 805-836.
- [Ni1] [16] K. Nishihara, J. Zhai *Asymptotic behaviors of solutions for time dependent damped wave equations*, J. Math. Anal. Appl. **360** (2009), 412-421.
- [Ni2] [17] J. Lin, K. Nishihara, J. Zhai  *$L^2$ -estimates of solutions for damped wave equations with space-time dependent damping term*, J. Differential Equations **248** (2010), 403-422.
- [NO] [18] M. Nakao, K. Ono, *Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations*, Math. Z. **214** (1993), 325-342.
- [NZ] [19] K. Nishihara, H. Zhao, *Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption*, J. Math. Anal. Appl. **313**(2) (2006), 598-610.
- [Ono1] [20] K. Ono, *Asymptotic behavior of solutions for semilinear telegraph equations*, J. Math. Tokushima Univ. **31**(1997), 11-22.
- [Ono2] [21] K. Ono, *Global solvability and  $L^p$  decay for the semilinear dissipative wave equations in four and five dimensions*, Funkcial. Ekvac. **49**(2) (2006), 215-233.
- [Ono3] [22] K. Ono, *Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations*, Discrete Contin. Dyn. Syst. **9** (2003), no. 3, 651-662.
- [TY] [23] G. Todorova, B. Yordnov, *Critical exponent for a nonlinear wave equation with damping*, J. Differential Equations **174** (2001), 464-489.
- [WY] [24] W. Wang, T. Yang, *The pointwise estimates of solutions for Euler equations with damping in multi-dimensions*, J. Differential Equations **173** (2001), 410-450.
- [Z] [25] Q. Zhang, *A blow-up result for a nonlinear wave equation with damping: the critical case*, C. R. Acad. Sci. Paris **333** (2001), 109-114.